

# LIOUVILLE THEOREM FOR THE FRACTIONAL LANE-EMDEN EQUATION IN UNBOUNDED DOMAIN

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**Abstract.** Our purpose of this paper is to study the nonexistence of nonnegative very weak solutions of

$$(-\Delta)^\alpha u = u^p + \nu \quad \text{in } \Omega, \quad u = g \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad (1)$$

where  $\alpha \in (0, 1)$ ,  $p > 0$ ,  $\Omega$  is a unbounded  $C^2$  domain in  $\mathbb{R}^N$  with  $N > 2\alpha$ ,  $g \in L^1(\mathbb{R}^N \setminus \Omega, \frac{dx}{1+|x|^{N+2\alpha}})$  nonnegative and  $\nu$  is a nonnegative Radon measure. We obtain that

(i) if  $\Omega \supseteq (\mathbb{R}^N \setminus \overline{B_{r_0}(0)})$  for some  $r_0 > 0$  and  $p < \frac{N}{N-2\alpha}$ , then problem (1) has no weak solutions.

(ii) if  $\Omega \supseteq \{x \in \mathbb{R}^N : x \cdot a > r_0\}$  for some  $r_0 \geq 0$ ,  $a \in \mathbb{R}^N$  and  $p < \frac{N+\alpha}{N-\alpha}$ , then problem (1) has no weak solutions. Here  $\frac{N+\alpha}{N-\alpha}$  is sharp for the nonexistence in the half space.

The above Liouville theorem could be applied to obtain nonexistence of classical solution of the fractional Lane-Emden equations

$$(-\Delta)^\alpha u = u^p \quad \text{in } \Omega, \quad u \geq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where  $\Omega = \mathbb{R}^N \setminus B_{r_0}(0)$  with  $r_0 > 0$  or  $\Omega = \mathbb{R}^{N-1} \times (0, +\infty)$ .

**Résumé.** Le but de cet article est d'étudier à la non-existence de solution nonnegative très faible de

$$(-\Delta)^\alpha u = u^p + \nu \quad \text{dans } \Omega, \quad u = g \quad \text{dans } \mathbb{R}^N \setminus \Omega, \quad (2)$$

où  $\alpha \in (0, 1)$ ,  $p > 0$ ,  $\Omega$  est un domaine de  $C^2$ , nonborné de  $\mathbb{R}^N$  avec  $N > 2\alpha$ ,  $g \in L^1(\mathbb{R}^N \setminus \Omega, \frac{dx}{1+|x|^{N+2\alpha}})$  nonnegative et  $\nu$  est une mesure de Radon nonnegative. On obtient alors

(i) si  $\Omega \supseteq (\mathbb{R}^N \setminus \overline{B_{r_0}(0)})$  pour certain  $r_0 > 0$  et  $p < \frac{N}{N-2\alpha}$ , alors le problème (2) n'a pas de solution faible.

(ii) si  $\Omega \supseteq \{x \in \mathbb{R}^N : x \cdot a > r_0\}$  pour certain  $r_0 \geq 0$ ,  $a \in \mathbb{R}^N$  et  $p < \frac{N+\alpha}{N-\alpha}$ , alors le problème (1) n'a pas de solution faible. Ici  $\frac{N+\alpha}{N-\alpha}$  est optimal pour la non-existence de solution dans le demi-espace.

Le théorème de Liouville précédent peut être appliqué pour montrer à la non-existence de solution classique de l'équation fractionnelle de Lane-Emden

$$(-\Delta)^\alpha u = u^p \quad \text{dans } \Omega, \quad u \geq 0 \quad \text{dans } \mathbb{R}^N \setminus \Omega,$$

où  $\Omega = \mathbb{R}^N \setminus B_{r_0}(0)$  avec  $r_0 > 0$  ou  $\Omega = \mathbb{R}^{N-1} \times (0, +\infty)$ .

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## 1. INTRODUCTION

Let  $\Omega$  be a  $C^2$  domain in  $\mathbb{R}^N$  satisfying that

$$(i) \Omega \supseteq \left( \mathbb{R}^N \setminus \overline{B_{r_0}(0)} \right) \quad \text{or} \quad (ii) \Omega \supseteq \{x \in \mathbb{R}^N : x \cdot a > 0\},$$

where  $r_0 > 0$  and  $a \in \mathbb{R}^N \setminus \{0\}$ . Our purpose of this paper is to study the nonexistence of nonnegative very weak solutions to the fractional Lane-Emden type equation

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + \nu & \text{in } \Omega, \\ u &= g & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.1}$$

where  $p > 0$ ,  $\nu$  is a nonnegative Radon measure in  $\Omega$ ,  $g$  is a nonnegative function in  $L^1(\mathbb{R}^N \setminus \Omega, \frac{dx}{1+|x|^{N+2\alpha}})$  and  $(-\Delta)^\alpha$  with  $\alpha \in (0, 1)$  is the fractional Laplacian defined in the principle value sense,

$$(-\Delta)^\alpha u(x) = c_{N,\alpha} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(x+z)}{|z|^{N+2\alpha}} dz,$$

here  $B_\epsilon(0)$  is the ball with radius  $\epsilon$  centered at the origin and  $c_{N,\alpha} > 0$  is the normalized constant, see [30] for details. In the particular case that  $\Omega = \mathbb{R}^N$  or  $\Omega = \mathbb{R}^N \setminus \{x_0\}$  for some point  $x_0 \in \mathbb{R}^N$ , the subjection:  $u = g$  in  $\mathbb{R}^N \setminus \Omega$  in (1.1) may be omitted.

It is known that the Liouville theorem plays a crucial role in deriving a priori estimates for solutions in PDE analysis and the nonexistence of entire solution to the second order differential equations has been studied for some decades. There is a large literature on the nonexistence of solutions for the problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \lim_{x \in \Omega, |x| \rightarrow +\infty} u(x) = 0. \tag{1.2}$$

Berestycki and Lions in [6] obtained the nonexistence results of (1.2) when  $\Omega = \mathbb{R}^N$ , Esteban [22] made use of a version of Maximum Principle to study the nonexistence of solutions of (1.2), when  $\Omega$  is a strip type domain. For general unbounded domain, the nonexistence result was derived by Esteban and Lions in [23]. The Liouville theorem has been extended to the fully nonlinear elliptic equations, see the references [3, 4, 5, 11, 12, 21, 32, 35], by developing the basic tools: the Maximum Principle and Hadamard Estimates.

During the last years, there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great interest in the model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars, see [8, 10, 36] and by important advances on the theory of nonlinear partial differential equations. The Liouville theorem of the nonlocal elliptic problems has been attracting the attentions, Felmer and Quaas in [27] extended the Hadamard estimate for the fractional Pucci's operator and obtained the Liouville theorem for the corresponding Lane-Emden equations.

As a typical nonlocal operator, the fractional Laplacian has been studied deeply, Z. Chen et al in [19, 20] derived the estimates for its Green's kernels by the stochastic method, W. Chen et al in [16, 17, 18] obtained the nonexistence of the entire solution for the fractional elliptic equations, M. Fall and T. Weth in [24, 25] obtained the nonexistence of positive solutions for a class of fractional semilinear elliptic equations in unbounded domains.

Recently, H. Chen et al in [14, 15] studied the fractional elliptic equation with Radon measures in bounded domain. In particular, the fractional Lane-Emden type equation

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + k\delta_0 & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{aligned} \tag{1.3}$$

has very weak solution when  $p < \frac{N}{N-2\alpha}$  and  $k > 0$  small, and has no very weak solution when  $p \geq \frac{N}{N-2\alpha}$ , where  $\Omega$  is a bounded domain containing the origin. One may ask if (1.3) has very weak solution when the domain  $\Omega$  is unbounded. Our motivation in this article is to clarify the existence and nonexistence when it involves unbounded domain, such as the whole domain, exterior domain and half space.

Before stating our main results, we make precise the notion of very weak solution used in this article. *A function  $u$  is said to be a very weak solution of (1.1) if  $u \in L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})$ ,  $|u|^p \in L^1_{loc}(\mathbb{R}^N)$  and*

$$\int_{\Omega} u(-\Delta)^{\alpha} \xi \, dx = \int_{\Omega} u^p \xi \, dx + \int_{\Omega} \xi \, d\nu - \int_{\Omega} \xi (-\Delta)^{\alpha} \tilde{g} \, dx, \quad \forall \xi \in C_c^{\infty}(\Omega), \quad (1.4)$$

where  $\tilde{g} = 0$  is the zero extension of  $g$  in  $\Omega$ ,  $C_c^{\infty}(\Omega)$  is the space of all the functions in  $C^{\infty}(\mathbb{R}^N)$  with compact support in  $\Omega$ . Although we set the test function  $\xi$  has compact support in  $\mathbb{R}^N$ , it follows by the nonlocal property of the fractional Laplacian that  $(-\Delta)^{\alpha} \xi(x)$  may have the decaying rate  $|x|^{-N-2\alpha}$  as  $|x| \rightarrow +\infty$ . This decaying at infinity requires that  $u, g \in L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})$ .

Now we state our first main results.

**Theorem 1.1.** *Assume that  $N > 2\alpha$ ,  $\nu$  is a nonnegative Radon measure in  $\Omega$ ,  $g$  is a nonnegative function in  $L^1(\mathbb{R}^N \setminus \Omega, \frac{dx}{1+|x|^{N+2\alpha}})$ .*

(i) *Assume that  $\Omega \supseteq (\mathbb{R}^N \setminus \overline{B_{r_0}(0)})$  for some  $r_0 > 0$ ,*

$$p \in \left(0, \frac{N}{N-2\alpha}\right). \quad (1.5)$$

*Then for any nonnegative measure  $\nu \neq 0$ , problem (1.1) has no nonnegative very weak solution.*

(ii) *Assume that  $\Omega \supseteq \{x \in \mathbb{R}^N : x \cdot a > r_0\}$  for some  $r_0 > 0$ ,  $a \in \mathbb{R}^N \setminus \{0\}$ ,*

$$p \in \left(0, \frac{N+\alpha}{N-\alpha}\right). \quad (1.6)$$

*Then for any nonnegative measure  $\nu \neq 0$ , problem (1.1) has no nonnegative very weak solution.*

The nonexistence of very weak solution in Theorem 1.1 is derived by contradiction. Let  $u$  be a nonnegative very weak solution of problem (1.1), the nonnegative source  $\nu$  would provide  $u$  an initial positive decay at infinity, then this decay will be reacted by the source nonlinearity  $u^p$ , until that  $u$  blows up everywhere. To our knowledge, our method is new and it could be applied in the Laplacian case, since it requires only the basic tools: the comparison principles (or Kato's inequality) and the estimates of the corresponding Green's kernel. This method, of course, is suitable in the classical sense. We say that  $u$  is a classical solution of

$$\begin{aligned} (-\Delta)^{\alpha} u &= u^p & \text{in } \Omega, \\ u &= g & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \quad (1.7)$$

if  $u$  is continuous in  $\Omega$  and satisfies the first equation in (1.7) pointwise in  $\Omega$ , where the function  $g$  is an outside source.

When  $\Omega$  is an exterior domain, i.e.  $\Omega = \mathbb{R}^N \setminus \overline{B_{r_0}(0)}$ , we have the following nonexistence of results.

**Corollary 1.1.** *Assume that  $p \in (0, \frac{N}{N-2\alpha})$  and  $u$  is a nonnegative classical solution of problem*

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \mathbb{R}^N \setminus B_{r_0}(0), \\ u &\geq 0 & \text{a.e. in } B_{r_0}(0). \end{aligned} \quad (1.8)$$

*Then  $u \equiv 0$  a.e. in  $\mathbb{R}^N$ .*

In the half space case, i.e.  $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$ , we have the following corollary.

**Corollary 1.2.** *Assume that  $p \in (0, \frac{N+\alpha}{N-\alpha})$  and  $u$  is a nonnegative classical solution of problem*

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \mathbb{R}_+^N, \\ u &\geq 0 & \text{a.e. in } \mathbb{R}^N \setminus \mathbb{R}_+^N, \end{aligned} \quad (1.9)$$

*where  $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times (0, \infty)$ . Then  $u \equiv 0$  a.e. in  $\mathbb{R}^N$ .*

Turning back to Theorem 1.1, we notice that in the case  $\Omega \supseteq \mathbb{R}^N \setminus \overline{B_{r_0}(0)}$ , problem (1.1) has no very weak solution when  $\nu = \delta_{\bar{x}}$  and  $p \geq \frac{N}{N-2\alpha}$ , because of the singularity at the point  $\bar{x} \in \Omega$ , here  $\frac{N}{N-2\alpha}$  is called by Serrin type exponent in the exterior domain or the whole domain. In the half space case  $\Omega = \mathbb{R}_+^N := \mathbb{R}^{N-1} \times (0, +\infty)$ , the Serrin type exponent is  $\frac{N+\alpha}{N-\alpha}$ , which is sharp for the nonexistence. In fact, problem (1.1) has a very weak solution under the hypotheses that  $\nu$  is a Dirac mass and  $\frac{N+\alpha}{N-\alpha} \leq p < \frac{N}{N-2\alpha}$ . Precisely, the existence result reads as follows.

**Theorem 1.2.** *Assume that  $k > 0$ ,  $\delta_{e_N}$  is the Dirac mass concentrated on the point  $e_N = (0, \dots, 0, 1)$  and*

$$p \in \left[ \frac{N+\alpha}{N-\alpha}, \frac{N}{N-2\alpha} \right). \quad (1.10)$$

*Then there exists  $k^* > 0$  such that for  $k \in (0, k^*)$ ,*

$$\begin{aligned} (-\Delta)^\alpha u &= u^p + k\delta_{e_N} & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}_+^N \end{aligned} \quad (1.11)$$

*admits a minimal positive solution  $u_k$ , which is a classical solution of*

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \mathbb{R}_+^N \setminus \{e_N\}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}_+^N. \end{aligned} \quad (1.12)$$

In contrast with the existence of positive solutions to (1.12), Chen, Fang and Yang in [16] obtained that the problem

$$\begin{aligned} (-\Delta)^\alpha u &= u^p & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}_+^N \end{aligned} \quad (1.13)$$

has only zero nonnegative solution under the hypotheses that

$$p > \frac{N}{N-2\alpha} \quad \text{and} \quad u \in L^{\frac{N(p-1)}{2\alpha}}(\mathbb{R}_+^N).$$

When  $1 < p < \frac{N-1+2\alpha}{N-1-2\alpha}$ , the nonexistence of positive bounded classical solution to (1.13) has been obtained independently in [24, 33]. These Liouville type theorems are derived by employing the method of moving planes. However, when it involves nontrivial nonnegative outside source, the method of moving planes is no longer valid and the critical exponent for the nonexistence reduces to  $\frac{N+\alpha}{N-\alpha}$ , see Corollary 1.2, and Theorem 1.2 provides an example showing the existence when  $p \geq \frac{N+\alpha}{N-\alpha}$ . In fact, let  $w(x) = u_k(x + 2e_N)$ , where  $u_k$  is the very

weak solution of (1.11), then  $w$  is a classical solution of (1.9) with nontrivial nonnegative outside source.

The paper is organized as follows. In Section 2, we show basic properties of the solutions to nonhomogeneous problem with nonzero outside source, the Integration by Parts formula, Comparison Principle. Section 3 is devoted to prove the nonexistence of nonnegative solutions to (1.1) and to prove the nonexistence in the classical setting. Finally, we prove the existence of very weak solutions of (1.8).

## 2. PRELIMINARY

Given a  $C^2$  domain  $\mathcal{O}$ , denote  $d_{\mathcal{O}}(x) = \text{dist}(x, \partial\mathcal{O})$ , denote by  $G_{\alpha, \mathcal{O}}$  the Green's function in  $\mathcal{O} \times \mathcal{O}$  and by  $\mathbb{G}_{\alpha, \mathcal{O}}[\nu]$  the very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu & \text{in } \mathcal{O}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O}, \end{aligned}$$

where  $\nu$  is a Radon measure in  $\mathcal{O}$ . In fact, for almost every  $x \in \mathcal{O}$ ,

$$\mathbb{G}_{\alpha, \mathcal{O}}[\nu](x) = \int_{\mathcal{O}} G_{\alpha, \mathcal{O}}(x, y) d\nu(y).$$

In what follows, we always denote by  $c_i$  the positive constant with  $i \in \mathbb{N}$ .

We first introduce the strong Comparison Principle.

**Lemma 2.1.** *Assume that  $\mathcal{D}$  is a  $C^2$  domain, functions  $f_1, f_2 \in C(\mathcal{D})$  satisfy  $f_2 \geq f_1$  in  $\mathcal{D}$  and  $g_1, g_2 \in C(\mathbb{R}^N \setminus \mathcal{D})$  satisfy  $g_2 \geq g_1$ .*

*Let  $u_i$  be the classical solutions of*

$$\begin{aligned} (-\Delta)^\alpha u &= f_i & \text{in } \mathcal{D}, \\ u &= g_i & \text{in } \mathbb{R}^N \setminus \mathcal{D} \end{aligned}$$

*with  $i = 1, 2$ , respectively. If*

$$\liminf_{x \rightarrow \partial\mathcal{D}} u_2(x) \geq \limsup_{x \rightarrow \partial\mathcal{D}} u_1(x)$$

*and for unbounded domain,*

$$\liminf_{|x| \rightarrow \infty} u_2(x) \geq \limsup_{|x| \rightarrow \infty} u_1(x),$$

*then*

$$u_2 \geq u_1 \quad \text{in } \mathcal{D}.$$

*Proof.* If  $\inf_{x \in \mathcal{D}} (u_2 - u_1)(x) < 0$ , then there exists a point  $x_0 \in \mathcal{D}$  such that  $(u_2 - u_1)(x_0) = \inf_{x \in \mathcal{D}} (u_2 - u_1)(x) = \text{essinf}_{x \in \mathbb{R}^N} (u_2 - u_1)(x)$ , which implies that

$$(-\Delta)^\alpha (u_2 - u_1)(x_0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{(u_2 - u_1)(x_0) - (u_2 - u_1)(y)}{|x_0 - y|^{N+2\alpha}} dy < 0.$$

However,

$$(-\Delta)^\alpha (u_2 - u_1)(x_0) = f_2(x_0) - f_1(x_0) \geq 0,$$

which is impossible.  $\square$

Next we introduce the weak Comparison Principles, which could be derived by the Kato's inequality in the fractional setting.

**Proposition 2.1.** [15, Proposition 2.4] *Assume that  $\mathcal{O}$  is a  $C^2$  bounded domain and  $f \in L^1(\mathcal{O}, d_{\mathcal{O}}^\alpha dx)$ , where  $d_{\mathcal{O}}(x) = \text{dist}(x, \partial\mathcal{O})$ . Then there exists a unique weak solution  $u$  to the problem*

$$\begin{aligned} (-\Delta)^\alpha u &= f & \text{in } \mathcal{O}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O}. \end{aligned} \quad (2.1)$$

Moreover, for any  $\xi \in C^{1,1}(\mathcal{O}) \cap C_0^\alpha(\mathcal{O})$ ,  $\xi \geq 0$ , we have that

$$\int_{\mathcal{O}} |u| (-\Delta)^\alpha \xi dx \leq \int_{\mathcal{O}} \xi \text{sign}(u) f dx \quad (2.2)$$

and

$$\int_{\mathcal{O}} u_+ (-\Delta)^\alpha \xi dx \leq \int_{\mathcal{O}} \xi \text{sign}_+(u) f dx. \quad (2.3)$$

We say that  $u_g$  is a very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu & \text{in } \mathcal{O}, \\ u &= g & \text{in } \mathbb{R}^N \setminus \mathcal{O}, \end{aligned} \quad (2.4)$$

if  $u \in L^1(\mathcal{O}, d_{\mathcal{O}}(x)^\alpha dx)$  and

$$\int_{\mathcal{O}} u_g (-\Delta)^\alpha \xi dx = \int_{\mathcal{O}} \xi d\nu - \int_{\mathcal{O}} \xi (-\Delta)^\alpha \tilde{g} dx, \quad \forall \xi \in C_c^\infty(\mathcal{O}), \quad (2.5)$$

where  $\nu$  is a bounded Radon measure in  $\mathcal{O}$ ,  $g \in L^1(\mathcal{O}^c, \frac{dx}{1+|x|^{N+2\alpha}})$ ,  $\tilde{g}$  is the zero extension of  $g$  in  $\mathcal{O}$ . When  $g = 0$  in  $\mathbb{R}^N \setminus \mathcal{O}$ , The authors in [15] showed that problem (2.4) admits a unique very weak solution  $u_0$ . When it involves the nonzero outside source  $g$ , the first difficulty is the Integration by Parts formula. From the definition of  $(-\Delta)^\alpha$ , we have that

$$(-\Delta)^\alpha \tilde{g}(x) = c_{N,\alpha} \int_{\mathbb{R}^N \setminus \mathcal{O}} \frac{g(y)}{|x-y|^{N+2\alpha}} dy, \quad \forall x \in \mathcal{O}. \quad (2.6)$$

**Lemma 2.2.** *Assume that  $\mathcal{O}$  is a bounded  $C^2$  domain,  $\nu$  is a bounded Radon measure,  $g \in L^1(\mathcal{O}^c, \frac{dx}{1+|x|^{N+2\alpha}})$  and  $u_0$  is a very weak solution of (2.4) with  $g = 0$  in  $\mathbb{R}^N \setminus \mathcal{O}$ .*

*Then (2.4) admits a unique very weak solution  $u_g$  satisfying that*

$$\text{if } g \geq 0, \text{ then } u_g \geq u_0$$

and

$$\text{if } g \leq 0, \text{ then } u_g \leq u_0.$$

*Proof.* Let  $\nu_n$  be a  $C^2$  sequence of functions converging to  $\nu$  in the dual sense of  $C(\bar{\mathcal{O}})$ . For  $\xi \in C_c^\infty(\mathcal{O})$ , there exists  $r_1 > 0$  such that

$$\text{supp } \xi \subset \mathcal{O}_1 := \{x \in \mathcal{O} : \text{dist}(x, \partial\mathcal{O}) > r_1\}.$$

Let  $u_n$  be the classical solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu_n & \text{in } \mathcal{O}_1, \\ u &= \tilde{g} & \text{in } \mathbb{R}^N \setminus \mathcal{O}_1. \end{aligned} \quad (2.7)$$

Since  $\tilde{g} = 0$  in  $\mathcal{O}$ , then  $(-\Delta)^\alpha \tilde{g} \in C_{loc}^1(\mathcal{O})$ . Let  $w_n = u_n - \tilde{g}$  and  $w_n$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu_n - (-\Delta)^\alpha \tilde{g} & \text{in } \mathcal{O}_1, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O}_1. \end{aligned}$$

Let  $v_n$  be the classical solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \nu_n & \text{in } \mathcal{O}_1, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O}_1. \end{aligned}$$

If  $g \geq 0$ , it follows by (2.6) that  $(-\Delta)^\alpha \tilde{g} \geq 0$ , and by Comparison Principle, we have that

$$v_n \leq u_n \quad \text{in } \mathbb{R}^N. \quad (2.8)$$

By the Integration by Parts formula, see [15, Lemma 2.2], we know that

$$\int_{\mathcal{O}} w_n (-\Delta)^\alpha \xi dx = \int_{\mathcal{O}} \xi \nu_n dx - \int_{\mathcal{O}} \xi (-\Delta)^\alpha \tilde{g} dx. \quad (2.9)$$

From [15, Proposition 2.2], it holds that

$$\begin{aligned} \|w_n\|_{M^{\frac{N}{N-2\alpha}}(\mathcal{O}_1)} &\leq c_1 \|\nu_n\|_{L^1(\mathcal{O}_1)} + c_1 \|(-\Delta)^\alpha \tilde{g}\|_{L^1(\mathcal{O}_1)} \\ &\leq c_2 \|\nu\|_{\mathfrak{M}(\mathcal{O})} + c_2 \|(-\Delta)^\alpha \tilde{g}\|_{L^1(\mathcal{O}_1)} \end{aligned}$$

where  $M^{\frac{N}{N-2\alpha}}(\mathcal{O}_1)$  is the Marcinkiewicz space with exponent  $\frac{N}{N-2\alpha}$  in  $\mathcal{O}_1$  and  $\mathfrak{M}(\mathcal{O})$  is the bounded Radon measure space in  $\mathcal{O}$ . By [15, Proposition 2.6], there exists  $u_g \in L^1(\mathcal{O})$  such that, up to some subsequence,

$$v_n \rightarrow u_0, \quad w_n \rightarrow u_g \quad \text{in } L^1(\mathcal{O}) \quad \text{as } n \rightarrow +\infty.$$

Passing the limit in (2.9) as  $n \rightarrow +\infty$ , we have that

$$\int_{\mathcal{O}} u_g (-\Delta)^\alpha \xi dx = \int_{\mathcal{O}} \xi d\nu - \int_{\mathcal{O}} \xi (-\Delta)^\alpha \tilde{g} dx.$$

It follows by (2.8) that

$$u_g \geq u_0 \quad \text{in } \mathbb{R}^N.$$

Now we prove the uniqueness. Assume that problem (2.4) has two solutions  $u_1, u_2$ , then  $w = u_1 - u_2$  is a very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= 0 & \text{in } \mathcal{O}, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \mathcal{O}. \end{aligned}$$

Then nonhomogeneous term is zero, of course, which is in  $L^1(\mathcal{O})$ , so by applying Proposition 2.1, we have that  $w \equiv 0$  a.e. in  $\mathcal{O}$ . The proof is complete.  $\square$

**Remark 2.1.** From Divergence theorem, the following identity holds

$$\int_{\mathcal{O}} (-\Delta) \xi(x) dx = 0, \quad \forall \xi \in C_c^\infty(\mathcal{O}).$$

In contrast with the Laplacian case, the corresponding identity for the fractional Laplacian reads

$$\int_{\mathcal{O}} (-\Delta)^\alpha \xi(x) dx = \int_{\mathcal{O}} \xi(x) \int_{\mathbb{R}^N \setminus \mathcal{O}} \frac{c_{N,\alpha}}{|x-y|^{N+2\alpha}} dy dx, \quad \forall \xi \in C_c^\infty(\mathcal{O}),$$

which could be obtained from (2.5) by the solution  $u \equiv 1$  of (2.4) taking  $\nu = 0$  and  $g = 1$  in  $\mathbb{R}^N \setminus \mathcal{O}$ .

**Corollary 2.1.** Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be  $C^2$  domains such that

$$\mathcal{O}_1 \subset \mathcal{O}_2.$$

Then

$$G_{\alpha, \mathcal{O}_1}(x, y) \leq G_{\alpha, \mathcal{O}_2}(x, y), \quad \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad x \neq y. \quad (2.10)$$

*Proof. Case 1:*  $\mathcal{O}_1$  is bounded. Since  $\mathcal{O}_1 \subset \mathcal{O}_2$ , then for fixed  $y \in \mathcal{O}_1$ ,  $G_{\alpha, \mathcal{O}_1}(\cdot, y)$ ,  $G_{\alpha, \mathcal{O}_2}(\cdot, y)$  are the solutions to

$$(-\Delta)^\alpha u = \delta_y \quad \text{in } \mathcal{O}_1,$$

subjecting to  $u = 0$  in  $\mathbb{R}^N \setminus \mathcal{O}_1$  and to  $u = G_{\alpha, \mathcal{O}_2}(\cdot, y)$  in  $\mathbb{R}^N \setminus \mathcal{O}_1$ , respectively. Then applying Lemma 2.2, we obtain that

$$G_{\alpha, \mathcal{O}_1}(x, y) \leq G_{\alpha, \mathcal{O}_2}(x, y), \quad \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad x \neq y.$$

For fixed  $y \in \mathbb{R}^N \setminus \mathcal{O}_1$ , obvious that  $G_{\alpha, \mathcal{O}_1}(\cdot, y) \equiv 0$  in  $\mathbb{R}^N$  and  $G_{\alpha, \mathcal{O}_2}(\cdot, y) \geq 0$  in  $\mathbb{R}^N \setminus \{y\}$ . Thus, (2.10) holds.

Case 2:  $\mathcal{O}_1$  is unbounded. For any  $y \in \mathcal{O}_1$ , let  $w = G_{\alpha, \mathcal{O}_1}(\cdot, y) - G_{\alpha, \mathcal{O}_2}(\cdot, y)$ , then  $w$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha w &= 0 \quad \text{in } \mathcal{O}_1 \setminus \{y\}, \\ w &\leq 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{O}_1. \end{aligned}$$

By the basic facts

$$\lim_{x \rightarrow y} G_{\alpha, \mathcal{O}_1}(x, y) |x - y|^{N-2\alpha} = \lim_{x \rightarrow y} G_{\alpha, \mathcal{O}_2}(x, y) |x - y|^{N-2\alpha} = c_3,$$

we derive that

$$\lim_{x \rightarrow y} w(x) |x - y|^{N-2\alpha} = 0,$$

thus, for any  $\epsilon$ , there exists  $r > 0$  such that  $w(x) \leq \epsilon G_{\alpha, \mathbb{R}^N}(x, y)$ ,  $\forall x \in B_\epsilon(0) \setminus \{y\}$ . We observe that

$$(-\Delta)^\alpha (w - \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y)) = 0 \quad \text{in } \mathcal{O}_1 \setminus B_\epsilon(0). \quad (2.11)$$

Since  $G_{\alpha, \mathbb{R}^N}(\cdot, y) > 0$  in  $\mathbb{R}^N \setminus \mathcal{O}_1$  and

$$\lim_{|x| \rightarrow +\infty} \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y) = 0 = \lim_{|x| \rightarrow +\infty} w(x)$$

so if

$$\sup_{x \in \mathbb{R}^N \setminus \{y\}} (w - \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y)) > 0,$$

there exists  $x_0 \in \mathcal{O}_1 \setminus B_\epsilon(0)$  such that

$$w(x_0) - \epsilon G_{\alpha, \mathbb{R}^N}(x_0, y) = \sup_{x \in \mathbb{R}^N \setminus \{y\}} (w - \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y)) > 0,$$

and from the definition of the fractional Laplacian, we obtain that

$$\begin{aligned} & (-\Delta)^\alpha (w - \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y)) (x_0) \\ &= c_{N, \alpha} \int_{\mathbb{R}^N} \frac{(w(x_0) - \epsilon G_{\alpha, \mathbb{R}^N}(x_0, z)) - (w(z) - \epsilon G_{\alpha, \mathbb{R}^N}(z, y))}{|x_0 - z|^{N-2\alpha}} dz > 0, \end{aligned}$$

which contradicts (2.11). This is to say that for any  $\epsilon > 0$ ,

$$w \leq \epsilon G_{\alpha, \mathbb{R}^N}(\cdot, y) \quad \text{in } \mathbb{R}^N \setminus \{y\}.$$

Therefore,  $w \leq 0$  in  $\mathbb{R}^N \setminus \{y\}$  and (2.10) holds. The proof is complete.  $\square$

Next we make a general estimate of the very weak solution of fractional equation with nonlinearity.



**Lemma 2.3.** Assume that  $\mathcal{D}$  is a  $C^2$  domain in  $\mathbb{R}^N$  (not necessary bounded),  $f : [0, +\infty) \rightarrow [0, +\infty)$  and  $\nu$  is a nonnegative Radon measure satisfying

$$\int_{\Omega} \frac{1}{1 + |y|^{N-2\alpha}} d\nu < +\infty. \quad (2.12)$$

Let  $u \geq 0$  be a very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= f(u) + \nu & \text{in } \mathcal{D}, \\ u &\geq 0 & \text{in } \mathbb{R}^N \setminus \mathcal{D}. \end{aligned} \quad (2.13)$$

Then

$$u \geq \mathbb{G}_{\alpha, \mathcal{D}}[\nu] \quad \text{a.e. in } \mathcal{D}.$$

*Proof.* By (2.12), we observe that  $\mathbb{G}_{\mathcal{D}, \alpha}[\nu]$  is well defined. Let  $D_n$  be a  $C^2$  bounded domain such that

$$\mathcal{D} \cap B_n(0) \subset D_n \subset \mathcal{D} \cap B_{n+1}(0) \quad \text{and} \quad \mathcal{D} = \bigcup_{n=1} D_n.$$

Let  $u_n$  be the positive very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u_n &= u^p + \nu & \text{in } D_n, \\ u_n &= 0 & \text{in } \mathbb{R}^N \setminus D_n. \end{aligned}$$

Since  $u \geq 0$ , then it follows by Corollary 2.1 that for any  $n$ ,

$$u \geq u_n \quad \text{a.e. in } \mathbb{R}^N. \quad (2.14)$$

Let  $v_n$  be the positive very weak solution of

$$\begin{aligned} (-\Delta)^\alpha v_n &= \nu & \text{in } D_n, \\ v_n &= 0 & \text{in } \mathbb{R}^N \setminus D_n. \end{aligned}$$

Then  $w_n := v_n - u_n$  is a weak solution of

$$\begin{aligned} (-\Delta)^\alpha w_n &= -u^p & \text{in } D_n, \\ w_n &= 0 & \text{in } \mathbb{R}^N \setminus D_n. \end{aligned}$$

From the Kato's inequality (2.3) with  $\xi$  the first eigenfunction of  $((-\Delta)^\alpha, D_n)$ , we have that

$$w_n \leq 0 \quad \text{a.e. in } D_n,$$

thus, together with (2.14), we have that

$$u \geq v_n \quad \text{a.e. in } \mathbb{R}^N. \quad (2.15)$$

Passing the limit as  $n \rightarrow \infty$ , we have that

$$u \geq \mathbb{G}_{\alpha, \mathcal{D}}[\nu].$$

The proof is complete.  $\square$

**Remark 2.2.** Under the hypotheses of Lemma 2.3, let  $\mu$  be a nonnegative Radon measure such that

$$\mu \leq \nu.$$

Then the nonnegative weak solution  $u$  of (2.13) satisfies

$$u \geq \mathbb{G}_{\alpha, \mathcal{D}}[\mu] \quad \text{a.e. in } \mathcal{D}.$$

Let  $\tau_0 < 0$  and  $\{\tau_j\}_j$  be the sequence generated by

$$\tau_j = 2\alpha + p\tau_{j-1} \quad \text{for } j = 1, 2, 3, \dots \quad (2.16)$$

**Lemma 2.4.** *Assume that*

$$p \in \left(0, 1 + \frac{2\alpha}{-\tau_0}\right), \quad (2.17)$$

*then  $\{\tau_j\}_j$  is an increasing sequence and there exists  $j_0 \in \mathbb{N}$  such that*

$$\tau_{j_0} \geq 0 \quad \text{and} \quad \tau_{j_0-1} < 0. \quad (2.18)$$

*Proof.* For  $p \in (0, 1 + \frac{2\alpha}{-\tau_0})$ , we have that

$$\tau_1 - \tau_0 = 2\alpha + \tau_0(p - 1) > 0$$

and

$$\tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0), \quad (2.19)$$

which imply that the sequence  $\{\tau_j\}_j$  is increasing. If  $p \geq 1$ , our conclusions are obvious. If  $p \in (0, 1)$ , we have that in the case that  $\tau_1 \geq 0$ , we are done, and in the case that  $\tau_1 < 0$ , it deduces from (2.19) that

$$\begin{aligned} \tau_j &= \frac{1 - p^j}{1 - p}(\tau_1 - \tau_0) + \tau_0 \\ &\rightarrow \frac{1}{1 - p}(\tau_1 - \tau_0) + \tau_0 = \frac{2\alpha}{1 - p} > 0, \quad \text{as } j \rightarrow +\infty, \end{aligned}$$

then there exists  $j_0 > 0$  satisfying (2.18).  $\square$

In the section 3, we shall apply lemma 2.4 with  $\tau_0 = 2\alpha - N$  when  $\Omega \supset (\mathbb{R}^N \setminus B_{r_0}(0))$ , with  $\tau_0 = \alpha - N$  when  $\Omega \supseteq \{x \in \mathbb{R}^N : x \cdot a > 0\}$ . Furthermore, for  $\tau_0 \in (-N, -N + 2\alpha]$ , it deduces by the fact  $\tau_{j_0-1} < 0$  that if  $j_0 \geq 2$ ,

$$\tau_0 + p\tau_{j_0-2} < -N.$$

### 3. NONEXISTENCE

We prove the nonexistence of very weak solution of (1.1) by contradiction. Assume that problem (1.1) admits a nonnegative solution  $u$ , we will obtain a contradiction from its decay at infinity.

**3.1. The whole domain or the exterior domain.** In the case that  $\Omega = \mathbb{R}^N$ , the Green's function is

$$G_{\alpha, \Omega}(x, y) = \frac{c_3}{|x - y|^{N-2\alpha}}, \quad \forall x, y \in \mathbb{R}^N, \quad x \neq y.$$

For the general exterior domain, the Green's kernel couldn't be given explicitly, but we can give the following estimate, which will play an important role in the derivation of the decay at infinity of the nonnegative solution  $u$  to problem (1.1).

**Lemma 3.1.** *Assume that  $N > 2\alpha$ ,  $\Omega \supseteq (\mathbb{R}^N \setminus B_{r_0}(0))$ . Let  $G_{\alpha, \Omega}$  be the Green's function of  $(-\Delta)^\alpha$  in  $\Omega \times \Omega$ , then there exists  $c_4 > 0$  such that for  $x, y \in \mathbb{R}^N \setminus B_{4r_0}(0)$ ,*

$$G_{\alpha, \Omega}(x, y) \geq \frac{c_4}{|x - y|^{N-2\alpha}}. \quad (3.1)$$

*Proof.* From the scaling property, see [19, (1.2)], for any  $l > 0$  and any bounded  $C^{1,1}$  domain  $\mathcal{O}$ , there holds

$$G_{\alpha, \mathcal{O}}(x, y) = l^{2\alpha-N} G_{\alpha, l\mathcal{O}}\left(\frac{x}{l}, \frac{y}{l}\right). \quad (3.2)$$

We may assume that  $r_0 = \frac{1}{2}$ . Fixed  $y \in \mathbb{R}^N \setminus B_2(0)$ , let  $\Gamma_y$  be the solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \delta_y - |y|^{2\alpha-N} \delta_0 \quad \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0. \end{aligned} \quad (3.3)$$

Then we have that

$$\Gamma_y(x) = \frac{c_3}{|x-y|^{N-2\alpha}} - \frac{c_3|y|^{2\alpha-N}}{|x|^{N-2\alpha}}, \quad \forall x \in \mathbb{R}^N \setminus \{y, 0\}. \quad (3.4)$$

Denote

$$A_y = \{x \in \mathbb{R}^N \setminus \{y, 0\} : \Gamma_y(x) \leq 0\}$$

and  $x \in A_y$  if and only if

$$|y-x| \geq |y||x|.$$

On the one hand, for  $x$  satisfying

$$|y|-|x| \geq |y||x|, \quad (3.5)$$

we have that  $x \in A_y$ . We observe that (3.5) is equivalent to

$$|x| \leq \frac{|y|}{|y|+1},$$

that is,

$$B_{\frac{|y|}{|y|+1}}(0) \subset A_y,$$

which implies that for any  $|y| \geq 2$ ,

$$B_{r_0}(0) \subset A_y.$$

On the other hand, for any  $x \in A_y$ , we have that

$$|y|+|x| \geq |y||x|,$$

that is,

$$|x| \geq \frac{|y|}{|y|-1}.$$

So for any  $|y| \geq 2$ ,

$$A_y \subset B_{\frac{|y|}{|y|-1}}(0) \subset B_2(0).$$

We see that  $G_{\alpha,\Omega}(\cdot, y)$  is the very weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= \delta_y \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ \lim_{|x| \rightarrow +\infty} u(x) &= 0. \end{aligned} \quad (3.6)$$

Since  $\mathbb{R}^N \setminus \Omega \subset B_{r_0}(0)$ , it follows by Corollary 2.1 that

$$G_{\alpha,\Omega}(\cdot, y) \geq \Gamma_y \quad \text{in } \mathbb{R}^N \setminus \{y\}. \quad (3.7)$$

It follows by (3.4) that for  $|x| \geq 2$ ,

$$|y||x| \geq \frac{|x|+|y|}{2} \geq \frac{|x-y|}{2},$$

which implies that

$$\Gamma_y(x) \geq (1-2^{2\alpha-N}) \frac{c_{N,\alpha}}{|x-y|^{N-2\alpha}}.$$

Thus, (3.1) holds. The proof is complete.  $\square$

**Lemma 3.2.** *Let  $\{\tau_j\}_j$  be defined by (2.16) with  $\tau_0 = 2\alpha - N$ ,  $\nu$  be a positive measure and  $u$  be a nonnegative solution of (1.1) verifying*

$$u(x) \geq c_j |x|^{\tau_j}, \quad \forall x \in \mathbb{R}^N \setminus B_{4r_0}(0)$$

*for some  $c_j > 0$  and  $j \leq j_0 - 2$ . Then for  $p \in (0, \frac{N}{N-2\alpha})$ , there exists  $c_{j+1} > 0$  such that*

$$u(x) \geq c_{j+1} |x|^{\tau_{j+1}}, \quad \forall x \in \mathbb{R}^N \setminus B_{4r_0}(0).$$

*Proof.* For  $r > \max\{1, 4r_0\}$ , let  $O_r = B_r(0) \setminus B_{4r_0}(0)$  and  $v_r$  be the unique solution of

$$\begin{aligned} (-\Delta)^\alpha v_r(x) &= c_j^p |x|^{\tau_j p} \chi_{O_r}(x), \quad \forall x \in \Omega, \\ v_r(x) &= 0, \quad \forall x \in \mathbb{R}^N \setminus \Omega, \\ \lim_{|x| \rightarrow \infty} v_r(x) &= 0, \end{aligned} \tag{3.8}$$

where  $\chi_{O_r} = 1$  in  $O_r$  and  $\chi_{O_r} = 0$  in  $\mathbb{R}^N \setminus O_r$ . By Lemma 2.3, we have that for any  $r > 1$ ,

$$u \geq v_r \quad \text{in } \mathbb{R}^N.$$

From (3.1), we have that

$$G_{\alpha, \Omega}(x, y) \geq \frac{c_4}{|x - y|^{N-2\alpha}}, \quad \forall x, y \in \mathbb{R}^N \setminus B_{4r_0}(0), x \neq y.$$

We observe that

$$\begin{aligned} v_r(x) &= \mathbb{G}_\alpha[c_j^p \cdot |x|^{\tau_j p} \chi_{O_r}](x) \\ &\geq c_4 c_j^p \int_{O_r} \frac{|y|^{\tau_j p}}{|x - y|^{N-2\alpha}} dy \\ &= c_4 c_j^p |x|^{\tau_{j+1}} \int_{O_{\frac{r}{|x|}}(0) \setminus O_{\frac{1}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} dz \\ &\rightarrow c_4 c_j^p |x|^{\tau_{j+1}} \int_{\mathbb{R}^N \setminus B_{\frac{4r_0}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} dz \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

where  $e_x = \frac{x}{|x|}$ ,  $\tau_j p < -2\alpha$  and for  $x \in \mathbb{R}^N \setminus B_{4r_0}(0)$ ,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{\frac{4r_0}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} dz &\geq \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} dz \\ &= \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|z|^{\tau_j p}}{|e_N - z|^{N-2\alpha}} dz. \end{aligned}$$

Let

$$c_{j+1} = c_4 c_j^p \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|z|^{\tau_j p}}{|e_N - z|^{N-2\alpha}} dz,$$

then

$$u(x) \geq c_{j+1} |x|^{\tau_{j+1}}, \quad \forall x \in \mathbb{R}^N \setminus B_{4r_0}(0).$$

The proof is complete.  $\square$

Now we are ready to prove Theorem 1.1(i).

**Proof of Theorem 1.1 (i).** By contradiction, we assume that (1.1) has a very weak solution  $u \geq 0$ . Since  $\nu \neq 0$ , there exists  $n_1 > 4r_0$  such that

$$B_{n_1}(0) \cap \text{supp} \nu \neq \emptyset,$$

then we have that

$$\mathbb{G}_{\alpha, B_{n_1}}[\nu \chi_{B_{n_1}(0)}] > 0 \quad \text{in } O_{n_1},$$

and by Lemma 2.3, we have that

$$u \geq \mathbb{G}_{\alpha, \Omega}[\nu] \geq \mathbb{G}_{\alpha, B_{n_1}}[\nu \chi_{B_{n_1}(0)}] \quad \text{in } \mathbb{R}^N.$$

Let  $\mu = \mathbb{G}_{\alpha, B_{n_1}}^p[\nu \chi_{B_{n_1}(0)}] > 0$  in  $B_{n_1}(0)$ , then

$$u \geq \mathbb{G}_{\alpha}[\mu].$$

For  $x \in \mathbb{R}^N \setminus B_{2n_1}(0)$ , we have that

$$\mathbb{G}_{\alpha, \Omega}[\mu](x) \geq c_5 \int_{B_{n_1}(0) \setminus B_{4r_0}(0)} \frac{\mu(y) dy}{|x - y|^{N-2\alpha}} \geq c_6 \|\mu\|_{L^1(B_{n_1}(0) \setminus B_{4r_0}(0))} |x|^{2\alpha-N}.$$

Thus, there exists  $c_0 > 0$  such that

$$u(x) \geq c_0 |x|^{\tau_0}, \quad \forall x \in \mathbb{R}^N \setminus B_{4r_0}(0) \quad (3.9)$$

with  $\tau_0 = 2\alpha - N < 0$ . Then it implies by Lemma 3.4 that for any  $j \leq j_0 - 1$ ,

$$u(x) \geq c_j |x|^{\tau_j}, \quad x \in \mathbb{R}^N \setminus B_{2r_0}(0), \quad (3.10)$$

where  $\{\tau_j\}_j$  is given by (2.16) and  $c_j > 0$ .

Let  $v_r$  the solution of (3.8) with  $j = j_0 - 1$ , we have that for any  $r > 8r_0$ ,

$$u(x) \geq v_r(x), \quad \forall x \in \mathbb{R}^N.$$

Then for any  $x \in B_{8r_0}(0) \setminus B_{4r_0}(0)$ ,  $y \in B_r(0) \setminus B_{8r_0}(0)$ , we have that  $|x - y| \leq 2|y|$  and

$$\begin{aligned} u(x) &\geq c_3 c_{j_0-1}^p \int_{B_r(0) \setminus B_{4r_0}(0)} \frac{|y|^{\tau_{j_0-1}p}}{|x - y|^{N-2\alpha}} dy \\ &\geq c_7 \int_{B_r(0) \setminus B_{4r_0}(0)} |y|^{2\alpha-N+\tau_{j_0-1}p} dy \\ &\geq \begin{cases} c_8 [r^{\tau_{j_0}} - (4r_0)^{\tau_{j_0}}] & \text{if } \tau_{j_0} > 0 \\ c_8 [\log r - \log(4r_0)] & \text{if } \tau_{j_0} = 0 \end{cases} \\ &\rightarrow \infty \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

which contradicts that  $u \in L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})$  from the definition of very weak solution of (1.1). The proof ends.  $\square$

**3.2. Half space.** We first recall the Green's estimate of the fractional Laplacian in half type space .

**Lemma 3.3.** Assume that  $N > \alpha$  and  $\Omega \supset \mathbb{R}_+^N$ , then there exists  $c_9 > 0$  such that

$$G_{\alpha, \Omega}(x, y) \geq \frac{c_9}{|x - y|^{N-2\alpha}} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^\alpha \right\}, \quad \forall x, y \in \mathbb{R}_+^N. \quad (3.11)$$

*Proof.* From Corollary 2.1, we have that

$$G_{\alpha, \Omega}(x, y) \geq G_{\alpha, \mathbb{R}_+^N}(x, y), \quad x, y \in \mathbb{R}_+^N, \quad x \neq y.$$

From [20, Corollary 1.4], there exists  $c_{10} > 1$  such that for any  $x, y \in \mathbb{R}_+^N$ ,  $x \neq y$ ,

$$\frac{1}{c_{10}} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^\alpha \right\} \leq G_{\alpha, \mathbb{R}_+^N}(x, y) |x - y|^{N-2\alpha} \leq c_{10} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^\alpha \right\}, \quad (3.12)$$

which implies (3.11). The proof is complete.  $\square$

Let  $\mathcal{C}_r$  be the cone

$$\mathcal{C}_r = \bigcup_{t>r} \{(x', t) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < t\}.$$

**Lemma 3.4.** *Let  $\{\tau_j\}_j$  be given by (2.16) with  $\tau_0 = \alpha - N$ ,  $\nu$  be a positive measure and  $u$  be a nonnegative solution of (1.1) satisfying*

$$u(x) \geq c_j |x|^{\tau_j}, \quad \forall x \in \mathcal{C}_1$$

*for some  $c_j > 0$  and  $j \leq j_0 - 2$ . Then for  $p \in (0, \frac{N+\alpha}{N-\alpha})$ , there exists  $c_{j+1} > 0$  such that*

$$u(x) \geq c_{j+1} |x|^{\tau_{j+1}}, \quad \forall x \in \mathcal{C}_1.$$

*Proof.* For  $r > 1$ , let  $O_r = \mathcal{C}_1 \cap B_r(0)$  and  $v_r$  be the unique solution of

$$\begin{aligned} (-\Delta)^\alpha v_r(x) &= c_j^p |x|^{\tau_j p} \chi_{O_r}(x), \quad \forall x \in \Omega, \\ v_r(x) &= 0, \quad \forall x \in \mathbb{R}^N \setminus \Omega, \\ \lim_{|x| \rightarrow +\infty} v_r(x) &= 0, \end{aligned} \tag{3.13}$$

where  $\chi_{O_r} = 1$  in  $O_r$  and  $\chi_{O_r} = 0$  in  $\mathbb{R}^N \setminus O_r$ . By Lemma 2.3, we have that for any  $r > 1$ ,

$$u \geq v_r \quad \text{in } \mathbb{R}^N.$$

From (3.11), we have that

$$\begin{aligned} v_r(x) &= \mathbb{G}_\alpha[c_j^p \cdot |x|^{\tau_j p} \chi_{O_r}](x) \\ &\geq c_9 c_j^p \int_{O_r} |y|^{\tau_j p} \frac{1}{|x-y|^{N-2\alpha}} \min \left\{ 1, \left( \frac{x_N y_N}{|x-y|^2} \right)^\alpha \right\} dy \\ &= c_9 c_j^p |x|^{\tau_{j+1}} \int_{O_{\frac{r}{|x|}}(0) \setminus O_{\frac{1}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} \min \left\{ 1, \left( \frac{z_N}{|e_x - z|^2} \right)^\alpha \right\} dz \\ &\rightarrow c_9 c_j^p |x|^{\tau_{j+1}} \int_{\mathcal{C}_1 \setminus B_{\frac{1}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} \min \left\{ 1, \left( \frac{z_N}{|e_x - z|^2} \right)^\alpha \right\} dz \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

where  $e_x = \frac{x}{|x|}$ ,  $\tau_j p < -2\alpha$  and for any  $x \in \mathcal{C}_1$ ,

$$\begin{aligned} &\int_{\mathcal{C}_1 \setminus B_{\frac{1}{|x|}}(0)} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} \min \left\{ 1, \left( \frac{z_N}{|e_x - z|^2} \right)^\alpha \right\} dz \\ &\geq \int_{\mathcal{C}_1} \frac{|z|^{\tau_j p}}{|e_x - z|^{N-2\alpha}} \min \left\{ 1, \left( \frac{z_N}{|e_x - z|^2} \right)^\alpha \right\} dz \\ &\geq \int_{\mathcal{C}_1} \frac{|z|^{\tau_j p}}{(1+|z|)^{N-2\alpha}} \frac{z_N^\alpha}{(1+|z|)^{2\alpha}} dz. \end{aligned}$$

Let

$$c_{j+1} = c_9 c_j^p \int_{\mathcal{C}_1} \frac{|z|^{\tau_j p}}{(1+|z|)^{N-2\alpha}} \frac{z_N^\alpha}{(1+|z|)^{2\alpha}} dz,$$

then

$$u(x) \geq c_{j+1} |x|^{\tau_{j+1}}, \quad \forall x \in \mathcal{C}_1.$$

The proof is complete.  $\square$

Now we are ready to prove Theorem 1.1(ii).

**Proof of Theorem 1.1 (ii).** By contradiction, we assume that (1.1) has a very weak solution  $u \geq 0$ .

We first claim that there exists  $c_0 > 0$  such that

$$u(x) \geq c_0|x|^{\alpha-N}, \quad \forall x \in \mathcal{C}_1. \quad (3.14)$$

Indeed, let  $\{O_n\}_n$  be a sequence of  $C^2$  domain such that

$$\Omega \cap B_n(0) \subset O_n \subset \Omega \cap B_{n+1}(0).$$

Let  $n_2 > 1$  such that

$$B_{n_2}(0) \cap \text{supp} \nu \neq \emptyset,$$

then we have that

$$\mathbb{G}_{\alpha, O_{n_2}}[\nu \chi_{B_{n_2}(0)}] > 0 \quad \text{in } O_{n_2}$$

and

$$u \geq \mathbb{G}_{\alpha, \Omega}[\nu \chi_{B_{n_2}(0)}] \geq \mathbb{G}_{\alpha, O_{n_2}}[\nu \chi_{B_{n_2}(0)}] \quad \text{in } \mathbb{R}^N.$$

Let  $\mu = \mathbb{G}_{\alpha, O_{n_2}}^p[\nu \chi_{B_{n_2}(0)}]$  and then

$$u \geq \mathbb{G}_{\alpha}[\mu].$$

For  $x \in \mathcal{C}_1$  and  $y \in O_{n_1} \cap \{z \in \mathbb{R}^N : z_N > 1\}$ , we have that  $|x - y| \leq |x| + |y| < (n_2 + 1)|x|$  and  $x_N > \frac{\sqrt{2}}{2}|x|$

$$\begin{aligned} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^{\alpha} \right\} &\geq \min \left\{ 1, \left( \frac{x_N}{(n_2 + 1)^2 |x|^2} \right)^{\alpha} \right\} \\ &\geq \frac{c_{11}}{1 + |x|^{\alpha}}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_{\alpha, \Omega}[\mu](x) &\geq c_{11} \int_{O_{n_2}} \frac{\mu(y) dy}{|x - y|^{N-2\alpha}} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^{\alpha} \right\} dy \\ &\geq c_{11} \|\mu\|_{L^1(O_{n_2})} |x|^{\alpha-N}, \end{aligned}$$

where  $c_{11} > 0$  depends on  $n_2$ .

From (3.14), we have that

$$u(x) \geq c_0|x|^{\tau_0}, \quad \forall x \in \mathcal{C}_1$$

with  $\tau_0 = \alpha - N < 0$ . Then it implies by Lemma 3.4 that for any  $j \leq j_0 - 1$ ,

$$u(x) \geq c_j|x|^{\tau_j}, \quad \forall x \in \mathcal{C}_1, \quad (3.15)$$

where  $\{\tau_j\}_j$  is given by (2.16) and  $c_j > 0$ .

Let  $v_r$  the solution of (3.13) with  $j = j_0 - 1$ , we have that for any  $r > 1$ ,

$$u(x) \geq v_r(x), \quad \forall x \in \mathbb{R}^N.$$

Then for any  $x \in \mathcal{C}_1 \setminus B_r(0)$  with  $r > 1$ ,  $y \in \mathcal{C}_1 \setminus B_{2|x|}(0)$ ,  $|x - y| \leq 2|y|$ , we have that

$$\begin{aligned} u(x) &\geq c_9 c_{j_0-1}^p \int_{O_r(0) \setminus B_1(0)} |y|^{\tau_{j_0}-1p} \frac{1}{|x-y|^{N-2\alpha}} \min \left\{ 1, \left( \frac{x_N y_N}{|x-y|^2} \right)^\alpha \right\} dy \\ &\geq c_{11} x_N^\alpha \int_{O_r(0) \setminus B_{2|x|}(0)} |y|^{-N+\tau_{j_0}-1p} y_N^\alpha dy \\ &\geq \begin{cases} c_{12} [r^{\tau_{j_0}} - (2|x|)^{\tau_{j_0}}] & \text{if } \tau_{j_0} > 0 \\ c_{12} [\log r - \log(2|x|)] & \text{if } \tau_{j_0} = 0 \end{cases} \\ &\rightarrow \infty \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

which contradicts that  $u \in L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})$  from the definition of very weak solution of (1.1). The proof ends.  $\square$

**3.3. Nonexistence in the classical setting.** In this subsection, we prove the nonexistence of classical solutions of semi-linear elliptic equations (1.8) and (1.9) by using the method in the proof of Theorem 1.1. The main difference is that we use strong Comparison Principle replacing the weak one.

**Proof of Corollary 1.1.** Since  $u \geq 0$  is a classical solution of (1.8), then if there exists one point  $x_0 \in \mathbb{R}^N \setminus B_{r_0}(0)$  such that  $u(x_0) = 0$ , then we have that

$$\int_{\mathbb{R}^N} \frac{u(y)}{|x_0 - y|^{N+2\alpha}} dy = 0,$$

which implies that

$$u \equiv 0.$$

So we may assume that  $u > 0$  in  $\Omega$  and let  $u_l(x) = u(l^{-1}x)$  for  $x \in \mathbb{R}^N$ ,  $l > 1$ , then  $u_l$  is a positive solution of

$$\begin{aligned} (-\Delta)^\alpha u_l &= l^{2\alpha} u_l^p \quad \text{in } \mathbb{R}^N \setminus B_{lr_0}(0), \\ u_l &\geq 0 \quad \text{in } B_{lr_0}(0). \end{aligned}$$

We see that the positive function  $w_l := \mathbb{G}_{\alpha, \mathbb{R}^N \setminus \overline{B_{2lr_0}(0)}}[(l^{2\alpha} - 1)u_l^p \chi_{B_{4lr_0}(0) \setminus B_{2lr_0}(0)}]$  is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha w_l &= l^{2\alpha} u_l^p \quad \text{in } \mathbb{R}^N \setminus \overline{B_{2lr_0}(0)}, \\ w_l &= 0 \quad \text{in } \overline{B_{2lr_0}(0)}, \\ \lim_{|x| \rightarrow +\infty} w_l(x) &= 0. \end{aligned} \tag{3.16}$$

The remainder of the proof is similar to the proof of Theorem 1.1 (i) just replacing the weak Comparison Principle by strong Comparison Principle, so we just sketch the proof. By strong Comparison Principle, we have that

$$u_l \geq w_l \quad \text{in } \mathbb{R}^N.$$

By Lemma 3.1 with  $\Omega = \mathbb{R}^N \setminus B_{2lr_0}(0)$ ,

$$w_l(x) \geq c_{13}|x|^{2\alpha-N}, \quad |x| > 2lr_0,$$

thus,

$$u_l \geq c_{13}|x|^{2\alpha-N}, \quad |x| > 2lr_0.$$

By Lemma 2.3 and repeat the argument of the proof of Theorem 1.1 (i) to obtain that

$$u_l(x) = +\infty \quad \text{for } |x| \geq 2lr_0,$$

which contradicts that  $u$  is a classical solution of (3.16).  $\square$



**Proof of Corollary 1.2.** Since  $u \geq 0$  is a classical solution of (1.9), then if there exists one point  $y_0 \in \Omega$  such that  $u(y_0) = 0$ , then we have that

$$\int_{\mathbb{R}^N} \frac{u(y)}{|x_0 - y|^{N+2\alpha}} dy = 0,$$

which implies that

$$u \equiv 0.$$

So we may assume that  $u > 0$  in  $\Omega$ . For  $l > 1$ , let  $u_l(x) = u(l^{-1}x)$  for  $x \in \mathbb{R}^N$ , then  $u_l$  is a positive solution of

$$\begin{aligned} (-\Delta)^\alpha u_l &= l^{2\alpha} u_l^p & \text{in } \mathbb{R}^{N-1} \times (0, +\infty), \\ u_l &\geq 0 & \text{in } \mathbb{R}^{N-1} \times (-\infty, 0]. \end{aligned} \quad (3.17)$$

The remaind is similar to the proof of Theorem 1.1 (ii) with  $p < \frac{N+\alpha}{N-\alpha}$ , and we omit here.  $\square$

#### 4. EXISTENCE IN THE SUPERCRITICAL CASE

To prove Theorem 1.2, the following estimate plays an important role in the construction of the upper bound in the procedure of finding the solution.

**Lemma 4.1.** *For  $p \in \left[\frac{N+\alpha}{N-\alpha}, \frac{N}{N-2\alpha}\right)$ , we have that*

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]] \leq c_{14} \mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \quad \text{in } \mathbb{R}_+^N. \quad (4.1)$$

*Proof.* From (3.12), we have that for  $x \in \mathbb{R}_+^N$ ,  $x \neq e_N$ ,

$$\frac{1}{c_{10}|x - e_N|^{N-2\alpha}} \frac{x_N^\alpha}{1 + |x|^{2\alpha}} \leq \mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}](x) \leq \frac{c_{10}}{|x - e_N|^{N-2\alpha}} \frac{x_N^\alpha}{1 + |x|^{2\alpha}}. \quad (4.2)$$

Our aim here is to prove (4.1) in  $\mathbb{R}_+^N$ , which is divided into  $D_1 := B_{\frac{1}{2}}(e_N)$ ,  $D_2 = \{z \in \mathbb{R}_+^N : z_N < \frac{1}{4}\}$ ,  $D_3 := \{z \in \mathbb{R}_+^N : z_N \geq \frac{1}{4}, |z| > 8\}$  and  $D_4 = \mathbb{R}_+^N \setminus (D_1 \cup D_2 \cup D_3)$ . Since  $D_4$  is compact and  $\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]]$  has no singularity and decaying, then (4.1) holds in  $D_4$ .

*Case 1:*  $x \in D_1$ . For  $x \in D_1$ , we observe that

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]](x) \leq c_{15} \int_{\mathbb{R}_+^N} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p} (1 + |y|)^{\alpha p}} dy,$$

where

$$\begin{aligned}
& \int_{B_2(0)} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\
& \leq \int_{B_2(0)} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^\alpha \right\} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{|y - e_N|^{(N-2\alpha)p}} dy \\
& \leq \int_{B_2(0)} \frac{c_{10}}{|(x - e_N) - y|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha)p}} dy \\
& = |x - e_N|^{2\alpha - (N-2\alpha)p} \int_{B_{\frac{2}{|x - e_N|}}(0)} \frac{c_{10}}{|e - z|^{N-2\alpha}} \frac{1}{|z|^{(N-2\alpha)p}} dz \\
& \leq |x - e_N|^{2\alpha - (N-2\alpha)p} \left[ \int_{B_1(e)} \frac{c_{10}}{|e - z|^{N-2\alpha}} dz + \int_{B_1(0)} \frac{c_{10}}{|z|^{(N-2\alpha)p}} dz \right. \\
& \quad \left. + \int_{B_{\frac{2}{|x - e_N|}}(0)} \frac{c_{10}}{1 + |z|^{(N-2\alpha)(p+1)}} dz \right] \\
& \leq c_{16} |x - e_N|^{2\alpha - (N-2\alpha)p} \left( 1 + |x - e_N|^{(N-2\alpha)p - 2\alpha} \right) \\
& = c_{16} |x - e_N|^{2\alpha - (N-2\alpha)p} + c_{15}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^N \setminus B_2(0)} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\
& \leq \int_{\mathbb{R}^N \setminus B_2(0)} \min \left\{ 1, \left( \frac{x_N y_N}{|x - y|^2} \right)^\alpha \right\} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{|y|^{(N-\alpha)p}} dy \\
& \leq \int_{\mathbb{R}^N} \frac{c_{10}}{1 + |y|^{(N-\alpha)(p+1)}} dy \leq c_{17},
\end{aligned}$$

here  $e_N = \frac{x - e_N}{|x - e_N|}$  and  $(N - \alpha)(p + 1) > N$ . Therefore, (4.1) holds for  $x \in D_1$ .

*Case 2:  $x \in D_2$ .* We note that

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]](x) \leq c_{17} \int_{\mathbb{R}_+^N} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y) y_N^\alpha}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{2\alpha p}} dy.$$

For  $x \in D_2$  satisfying  $|x| \geq \frac{1}{2}$ , let  $D_x = \{z \in \mathbb{R}_+^N : z_N > 2x_N\}$ , then we have that

$$\begin{aligned}
& \int_{D_x} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\
& \leq x_N^\alpha \int_{D_x} \frac{y_N^\alpha}{1 + |x - y|^{2\alpha}} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{|y - e_N|^{(N-\alpha)p}} dy \\
& \leq x_N^\alpha |x|^{(\alpha-N)(p+1)} \int_{\mathbb{R}_+^N} \frac{c_{10}}{|e_x - z|^{N-\alpha}} \frac{1}{|z - \frac{e_N}{|x|}|^{(N-\alpha)p}} dy \\
& \leq c_{18} x_N^\alpha |x|^{(\alpha-N)(p+1)}
\end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^N \setminus D_x} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy &\leq \int_{\mathbb{R}^N \setminus D_x(0)} \frac{x_N^\alpha y_N^\alpha}{1 + |x - y|^{2\alpha}} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{1 + |y|^{(N-\alpha)p}} dy \\ &\leq c_{19} x_N^\alpha |x|^{(\alpha-N)(p+1)}, \end{aligned}$$

where  $(\alpha - N)(p + 1) < -N$ .

For  $x \in D_2$  satisfying  $|x| < \frac{1}{2}$ , we have that

$$\begin{aligned} &\int_{D_x} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\ &\leq x_N^\alpha \int_{D_x} \frac{y_N^\alpha}{1 + |x - y|^{2\alpha}} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{|y - e_N|^{(N-\alpha)p}} dy \\ &\leq c_{20} x_N^\alpha \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^N \setminus D_x} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy &\leq \int_{\mathbb{R}^N \setminus D_x(0)} \frac{x_N^\alpha y_N^\alpha}{1 + |x - y|^{2\alpha}} \frac{1}{|x - y|^{N-2\alpha}} \frac{1}{1 + |y|^{(N-\alpha)p}} dy \\ &\leq c_{21} x_N^\alpha. \end{aligned}$$

Therefore, (4.1) holds for  $x \in D_2$ .

*Case 3:*  $x \in D_3$ . We see that

$$\begin{aligned} &\int_{\mathbb{R}_+^N} \frac{G_{\alpha, \mathbb{R}_+^N}(x, y)}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\ &\leq \int_{\mathbb{R}_+^N} \frac{c_{10}}{|x - y|^{N-2\alpha}} \frac{1}{|y - e_N|^{(N-2\alpha)p}(1 + |y|)^{\alpha p}} dy \\ &= c_{22} |x|^{2\alpha - (N-\alpha)p} \int_{\mathbb{R}_+^N} \frac{c_{10}}{|e_x - z|^{N-2\alpha}} \frac{1}{|z - \frac{e_N}{|x|}|^{(N-2\alpha)p}(|x|^{-1} + |z|)^{\alpha p}} dz \\ &\leq c_{22} |x|^{2\alpha - (N-\alpha)p} \left[ \int_{B_{\frac{1}{2}}(e_N)} \frac{c_{10}}{|e_N - z|^{N-2\alpha}} dz + \int_{B_{\frac{1}{2}}(\frac{e_N}{|x|})} |z - \frac{e_N}{|x|}|^{(N-2\alpha)p} dz \right. \\ &\quad \left. + \int_{B_{\frac{2}{|x-e_N|}}(0)} \frac{c_{10}}{1 + |z|^{(N-\alpha)(p+1)-\alpha}} dz \right] \\ &\leq c_{23} |x|^{2\alpha - (N-\alpha)p}, \end{aligned}$$

where  $(N - \alpha)(p + 1) - \alpha > N$ . Since  $2\alpha - (N - \alpha)p \leq \alpha - N$ , then (4.1) holds for  $x \in D_3$ . The proof ends.  $\square$

**Proof of Theorem 1.2.** We first define the iterating sequence

$$v_0 = k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] > 0$$

and

$$v_n = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[v_{n-1}^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}].$$

Observing that

$$v_1 = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[(kv_0)^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] > v_0$$

and assuming that

$$v_{n-1} \geq v_{n-2} \quad \text{in} \quad \mathbb{R}_+^N \setminus \{e_N\},$$

we deduce that

$$v_n = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[v_{n-1}^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \geq \mathbb{G}_{\alpha, \mathbb{R}_+^N}[v_{n-2}^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] = v_{n-1}.$$

Then the sequence  $\{v_n\}_n$  is increasing with respect to  $n$ . Moreover, we have that

$$\int_{\mathbb{R}_+^N} v_n (-\Delta)^\alpha \xi \, dx = \int_{\mathbb{R}_+^N} v_{n-1}^p \xi \, dx + k\xi(e_N), \quad \forall \xi \in C_c^\infty(\mathbb{R}_+^N). \quad (4.3)$$

We next build an upper bound for the sequence  $\{v_n\}_n$ . For  $t > 0$ , denote

$$w_t = tk^p \mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \leq (c_{14}tk^p + k)\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}], \quad (4.4)$$

where  $c_{14} > 0$  is from Lemma 4.1, then

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[w_t^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \leq (c_{14}tk^p + k)^p \mathbb{G}_{\alpha, \mathbb{R}_+^N}[\mathbb{G}_{\alpha, \mathbb{R}_+^N}^p[\delta_{e_N}]] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \leq w_t,$$

if

$$(c_{14}tk^p + k)^p \leq tk^p,$$

that is,

$$(c_{14}tk^{p-1} + 1)^p \leq t. \quad (4.5)$$

Let  $k_p = \left(\frac{1}{c_{14}p}\right)^{\frac{1}{p-1}} \frac{p-1}{p}$  and  $t_p = \left(\frac{p}{p-1}\right)^p$ , then if  $k \leq k_p$  and  $t = t_p$ , (4.5) holds. Hence, by the definition of  $w_{t_p}$ , we have  $w_{t_p} > v_0$  and

$$v_1 = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[v_0^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0] < \mathbb{G}_{\alpha, \mathbb{R}_+^N}[w_{t_p}^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0] = w_{t_p}.$$

Inductively, we obtain that

$$v_n \leq w_{t_p} \quad (4.6)$$

for all  $n \in \mathbb{N}$ . Therefore, the sequence  $\{v_n\}_n$  converges. Let  $u_k := \lim_{n \rightarrow \infty} v_n$ . By (4.3), we have that  $u_k$  is a very weak solution of (1.8).

We claim that  $u_k$  is the minimal solution of (1.1), that is, for any nonnegative solution  $u$  of (1.8), we always have  $u_k \leq u$ . Indeed, there holds

$$u = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[u^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0] \geq v_0,$$

then

$$u = \mathbb{G}_{\alpha, \mathbb{R}_+^N}[u^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0] \geq \mathbb{G}_{\alpha, \mathbb{R}_+^N}[v_0^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0] = v_1.$$

We may show inductively that

$$u \geq v_n$$

for all  $n \in \mathbb{N}$ . The claim follows.

Similarly, if problem (1.8) has a nonnegative solution  $u$  for  $k_1 > 0$ , then (1.8) admits a minimal solution  $u_k$  for all  $k \in (0, k_1]$ . As a result, the mapping  $k \mapsto u_k$  is increasing. So we may define

$$k^* = \sup\{k > 0 : (1.1) \text{ has minimal solution for } k\}$$

and we have that

$$k^* \geq k_p.$$

*Regularity of the very weak solution of (1.8).* Let  $u$  be a very weak solution of (1.8), take  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_N) \in \mathbb{R}_+^N \setminus \{e_N\}$  and  $r = \frac{1}{4} \min\{|\bar{x} - e_N|, \bar{x}_N\}$ , then

$$\begin{aligned} u &= \mathbb{G}_{\alpha, \mathbb{R}_+^N}[u^p] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}] \\ &= \mathbb{G}_{\alpha, \mathbb{R}_+^N}[u^p \chi_{B_r(\bar{x})}] + \mathbb{G}_{\alpha, \mathbb{R}_+^N}[u^p \chi_{\mathbb{R}_+^N \setminus B_r(\bar{x})}] + k\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_{e_N}], \end{aligned}$$

where  $\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0]$  is  $C_{loc}^\infty(\mathbb{R}_+^N \setminus \{e_N\})$ . To be convenient, we write  $B_i = B_{2^{-i}r}(\bar{x})$ . For  $x \in B_i$ , we have that

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{\mathbb{R}_+^N \setminus B_{i-1}} u^p](x) = \int_{\mathbb{R}_+^N \setminus B_{i-1}} u(y)^p G_{\alpha, \mathbb{R}_+^N}(x, y) dy,$$

then, for some  $C_i > 0$ , we have that

$$\|\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{\mathbb{R}_+^N \setminus B_i} u^p]\|_{C^2(B_{i-1})} \leq C_i \|u^p\|_{L^1(B_{2r}(\bar{x}))} \quad (4.7)$$

and for some constant  $\tilde{c}_i > 0$  depending on  $i$ , we obtain that

$$\|\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\delta_0]\|_{C^2(B_{i-1})} \leq \tilde{c}_i |\bar{x}|^{2\alpha-N}. \quad (4.8)$$

By Proposition 2.2 in [14], we have that  $u^p \in L^{q_0}(B_{2r_0}(\bar{x}))$  with  $q_0 = \frac{1}{2}(1 + \frac{1}{p} \frac{N}{N-2\alpha}) > 1$  and then

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{B_{2r}(\bar{x})} u^p] \in L^{p_1}(B_{2r}(\bar{x})) \quad \text{with } p_1 = \frac{Nq_0}{N-2\alpha q_0}.$$

Similarly,

$$u^p \in L^{q_1}(B_r(\bar{x})) \quad \text{with } q_1 = \frac{p_1}{p}$$

and

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{B_r(\bar{x})} u^p] \in L^{p_2}(B_r(\bar{x})) \quad \text{with } p_2 = \frac{Nq_1}{N-2\alpha q_1}.$$

Let  $q_i = \frac{p_i}{p}$  and  $p_{i+1} = \frac{Nq_i}{N-2\alpha q_i}$  if  $N-2\alpha q_i > 0$ . Then we obtain inductively that

$$u^p \in L^{q_i}(B_i) \quad \text{and} \quad \mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{B_i} u^p] \in L^{p_{i+1}}(B_i).$$

We may verify that

$$\frac{q_{i+1}}{q_i} = \frac{1}{p} \frac{N}{N-2\alpha q_i} > \frac{1}{p} \frac{N}{N-2\alpha q_1} > 1.$$

Therefore,  $\lim_{i \rightarrow +\infty} q_i = +\infty$ , so there exists  $i_0$  such that  $N-2\alpha q_{i_0} > 0$ , but  $N-2\alpha q_{i_0+1} < 0$ , then we deduce that

$$\mathbb{G}_{\alpha, \mathbb{R}_+^N}[\chi_{B_{i_0}} u^p] \in L^\infty(B_{i_0}).$$

As a result, we obtain that

$$u \in L^\infty(B_{i_0}).$$

By regularity results in [34], we know from (4.8) that  $u$  is Hölder continuous in  $B_{i_0}$  and so is  $u^p$ . Then  $u$  is a classical solution of (1.1).  $\square$

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## REFERENCES

- [1] S. Alarcón, M. Burgos-Pérez, Á. Garca-Melián and A. Quaas, Nonexistence results for elliptic equations with gradient terms, *J. Diff. Eq.* 260(1), 758-780 (2016).
- [2] S.N. Armstrong, B. Sirakov, C.K. Smart, Fundamental solutions of homogeneous fully nonlinear elliptic equations, *Comm. Pure Appl. Math.* 64(6), 737-777 (2011).
- [3] S.N. Armstrong, B. Sirakov, Sharp Liouville results for fully nonlinear equations with power-growth nonlinearities, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 10(5), 711-728 (2011).
- [4] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems, *Topol. Methods Nonlinear Anal.* 4(1), 59-78 (1994).
- [5] H. Berestycki, F. Hamel and L. Rossi, Liouville type results for semilinear elliptic equations in unbounded domains, *Ann. Mat.* 186(3), 467-507 (2007).

- [6] H. Berestycki and P. Lions, Existence of solutions for nonlinear scalar field equations. Part I: The ground state, *Arch. Rational Meth. Anal.* 82, 313-345 (1983).
- [7] M. Bidaut-Véron and L. Véron, Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.* 106, 489-539 (1991).
- [8] J. Bouchard and A. Georges, Anomalous diffusion in disordered media: statistical mechanics, models and physical applications, *Phys. Rep.* 195, (1990).
- [9] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42, 271-297 (1989).
- [10] L. Caffarelli and L. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math.* 171, 1903-1930 (2010).
- [11] I. Capuzzo-Dolcetta, A. Cutrì, Hadamard and Liouville type results for fully nonlinear partial differential inequalities, *Comm. Contemp. Math.* 5, 435-448 (2003).
- [12] H. Chen and P. Felmer, On Liouville type theorems for fully nonlinear elliptic equations with gradient term, *J. Diff. Eq.* 255(8), 2167-2195 (2013).
- [13] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *Annales de l'Institut Henri Poincaré (C)* 32, 1199-1228 (2015).
- [14] H. Chen and A. Quaas, Classification of isolated singularities of nonnegative solutions to fractional semi-linear elliptic equations and the existence results, *ArXiv:1509.05836* (2015).
- [15] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *J. Differential equations* 257(5), 1457-1486 (2014).
- [16] W. Chen, Y. Fang and Y. Ray, Liouville theorems involving the fractional Laplacian on a half space, *Adv. Math.* 274, 167-198 (2015).
- [17] W. Chen and Y. Fang, A Liouville type theorem for poly-harmonic Dirichlet problem in a half space, *Adv. Math.* 229, 2835-2867 (2012).
- [18] W. Chen, X. Cui, Z. Yuan and R. Zhuo, A liouville theorem for the fractional laplacian, *arXiv:1401.7402* (2014).
- [19] Z. Chen and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann.* 312, 465-501 (1998).
- [20] Z. Chen and J. Tokle, Global heat kernel estimates for fractional Laplacians in unbounded open sets, *Probability theory and related fields* 149, 373-395 (2011).
- [21] A. Cutrì, F. Leoni, On the Liouville Property for fully nonlinear equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17(2), 219-245 (2000).
- [22] M. Esteban, Nonlinear elliptic problems in strip-like domains: symmetry of positive vortex rings, *Non-linear Analysis: Theory, Methods & Applications* 7, 365-379 (1983).
- [23] M. Esteban and P. Lions, Existence and non-existence results for semilinear elliptic problems in unbounded domains, *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 93, 1-14 (1982).
- [24] M. Fall and T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half-space, *Comm. Cont. Math.* 18(1), 1-25 (2016).
- [25] M. Fall and T. Weth, Nonexistence results for a class of fractional elliptic boundary values problems, *J. Funct. Anal.* 263, 2205-2227 (2012).
- [26] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of  $\mathbb{R}^N$ , *J. Math. Pures Appl.* 87, 537-561 (2007).
- [27] P. Felmer, A. Quaas, Fundamental solutions and Liouville type theorems for nonlinear integral operators, *Adv. Math.* 226(3), 2712-2738 (2011).
- [28] B. Gidas and J. Spruck, Global and local behaviour of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 34, 525-598 (1981).
- [29] N. Korevaar, R. Mazzeo, F. Pacard, and R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, *Invent. Math.* 135, 233-272 (1999).
- [30] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136, 521-573 (2012).
- [31] X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.* 337, 549-590 (1993).
- [32] A. Quaas, B. Sirakov, Existence and nonexistence results for fully nonlinear elliptic systems, *Indiana Univ. Math. J.* 58(2), 751-788 (2009).
- [33] A. Quaas and A. Xia, Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space, *Calculus of Variations and Partial Differential Equations* 52, 641-659 (2015).

- [34] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl.* 101(3), 275-302 (2014).
- [35] L. Rossi, Non-existence of positive solutions of fully nonlinear elliptic equations in unbounded domains, *Comm. Pure and Appl. Anal.* 7, 125-141 (2008).
- [36] V. Tarasov and G. Zaslavsky, Fractional dynamics of systems with long-range interaction, *Commun. Nonlinear Sci. Numer. Simul.* 11, 885-889 (2006).
- [37] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, Vol. I, 593-712, *Handb. Differ. Equ. North-Holland, Amsterdam* (2004).